Equivalent Deterministic Inputs for Random Processes

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Given a linear time-invariant multivariable system with a prescribed set of stationary random inputs, we wish to determine the correlations among the outputs. A theorem is presented that permits this determination to be made in the time domain by integrating the response to certain deterministic transient inputs derived from the spectrum matrix of the prescribed random inputs. The method is especially useful when a mathematical model of the system is not available. It is tested for a two-input system and is shown to give the same numerical results as integrating the response spectra.

Nomenclature

$C_{nn}(\tau), C_{rr}(\tau)$	= correlation matrices
$F[\ ]$	= Fourier transform
$G(s)$	= transfer function matrix
$n(t)$	= stationary random input
$r(t)$	= random response vector
$u(t)$	= deterministic function
$U(\omega)$	= Fourier transform of the equivalent deterministic function
W	= matrix $[U_1, \ldots, U_N]$
$w(t)$	= white noise input
$y(t)$	= response to deterministic inputs
σ	= root mean square
$\phi_{nn}(\omega)$	= spectral density matrix of random input
$\phi_{rr}(\omega)$	= spectral density matrix of output state variable
ω	= frequency, rad/s
$\langle\ \rangle$	= ensemble average
$(\ )^*$	= complex conjugate matrix
$(\ )^H$	= Hermitian transpose matrix
$(\ )^T$	= transpose matrix

I. Introduction

THIS paper is concerned with the response of stable linear-invariant systems to multiple random inputs. Its principal objective is to generalize the equivalent deterministic technique of Ref. 1, which treated only a single input, to multiple inputs and outputs.

The engineering problem that motivated us is the flight of airplanes in atmospheric turbulence,[2-4] and although the paper is slanted to that application, we believe that the method presented herein adds another useful tool to the kit available to the modern systems analyst. It is not immediately clear when the use of equivalent deterministic inputs should be preferred to other methods of calculating responses. We think it is more economical of computing time than calculating response spectra from input spectra and the matrix of system transfer functions [Eq. (1)]. We also think it is clearly better than any other method of finding the matrix of response covariances (or correlations) of a given existing system for which no mathematical model is available since it does not first require "system identification." A point we would emphasize is that the equivalent inputs are a set of deterministic transients that are fixed by the spectrum matrix of the random input, and do not depend on the driven system (except insofar as the system influences the inputs). Thus, in the context of the aircraft industry, once a particular model of atmospheric turbulence has been adopted (say, Dryden's or von Kármán's[5]) and a representation of the airplane is chosen (e.g., the four-point model[5]), then the equivalent inputs can be calculated, once and for all, for each size of airplane. They can subsequently be used with appropriate time and amplitude scaling for any flight condition or airplane configuration. (Size of the airplane is a parameter in the four-point model—changing size changes the input spectra and the equivalent transient inputs.)

Before proceeding with the details of the present development, we think it worthwhile to present a review of the existing methods for calculating response statistics.

II. Review of Existing Methods

The ultimate goal of random process analysis is frequently to calculate probability of failure, return period, lifetime, signal-to-noise ratio, ride comfort, fatigue damage, etc. When the random process is assumed to be Gaussian (a very common assumption), then all the above results can be obtained for any particular response variable r_k (which may be a linear combination of the system state variables) from the two quantities $\langle r_k^2 \rangle$ and $\langle \dot{r}_k^2 \rangle$. Thus the goal of a response calculation is frequently simply to find these two items. More generally, one might have reason to want the whole output correlation matrix $C_{rr}(\tau)$ or spectrum matrix $\phi_{rr}(\omega)$ from which the above two mean-square values can be deduced. Since C and ϕ are a Fourier integral pair, a knowledge of one implies the other.

The method used to find the desired response quantities depends on whether or not there is a mathematical model of the system, and, if so, what form the model takes.

Solution Using the Transfer Function

When the mathematical model consists of the transfer function matrix $G(i\omega)$ the spectra of the response $r(t)$ and the input $n(t)$ are related by[6]

$$\Phi_{rr}(\omega) = G(i\omega)\Phi_{nn}(\omega)G^H(i\omega) \tag{1}$$

The desired mean-squares are then given by the diagonal elements of

$$\langle rr^T \rangle = \int_{-\infty}^{\infty} \Phi_{rr}(\omega)\,\mathrm{d}\omega \tag{2}$$

and

$$\langle \dot{r}\dot{r}^T \rangle = \int_{-\infty}^{\infty} \omega^2 \Phi_{rr}(\omega)\,\mathrm{d}\omega \tag{3}$$

Submitted June 3, 1982; revision received July 5, 1983.

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In the aerospace application, when the system may have many lightly damped modes (sharp peaks in $|G|$) the accurate evaluation of Eqs. (2) and (3) can present some difficulty.

Solution Using the Impulse Response

The matrix of impulse responses, $H(t)$, is the Fourier transform of $G(i\omega)$, and yields the response by convolution with the input. In terms of H, the response covariance matrix can be written as

$$C_{rr}(\tau) \triangleq \langle r(t+\tau) r^T(\tau) \rangle$$

$$= \int_0^\infty \int_0^\infty H(\alpha) C_{nn}(\beta - \alpha + \tau) H^T(\beta) \mathrm{d}\alpha \mathrm{d}\beta \tag{4}$$

The desired mean-square values are given by $C_{rr}(0)$.

Solution Using the System Matrices

The response and input can always be related by a first-order differential equation of the form

$$\dot{r} = Ar + Bn \tag{5}$$

In fact, the differential equation for the state vector x of an airplane with control vector u, and subjected to a turbulence input g is of the form[5]

$$\dot{x} = Ax + Bu + C_1 g + C_2 \dot{g} \tag{6}$$

It is shown in Ref. 5 how to eliminate the $\dot{g}$ term from Eq. (6) to yield the canonical form without derivatives on the right-hand side. From Eq. (5) a differential equation can readily be derived for the propagation of the covariance.[9]

$$R(t) \triangleq \langle r(t) r^T(t) \rangle \triangleq C_{rr}(0) \tag{7}$$

That equation is

$$\dot{R} = AR + RA^T + B\langle n(t) r^T(t) \rangle + \langle r(t) n^T(t) \rangle B^T \tag{8}$$

When a steady state has been reached, as $t \to \infty$, Eq. (8) yields

$$O = AR + RA^T + BC_{nr}(0) + C_{nr}^T(0) B^T \tag{9}$$

where

$$C_{nr}(\tau) = \langle n(t+\tau) r^T(t) \rangle; \quad t \to \infty$$

Equation (9) can be used to compute the desired covariance matrix R. However, to do so, one needs as well the input-output covariance $C_{nr}(0)$. This can be calculated from the relation

$$C_{nr}^T(0) = C_{rn}(0) = \int_0^\infty e^{A\tau} B C_{nn}(-\tau) \mathrm{d}\tau \tag{10}$$

It should be noted that the above solution does *not* require that the input be white noise (in contradistinction to the treatment in Ref. 9). The mean-square values $\langle r_k^2 \rangle$ are the diagonal elements of R. To obtain derivatives of responses when needed, these can simply be added to the response vector, e.g.,

$$r_{k+1} = \dot{r}_k$$

The White Noise Case

A fairly common practice is to assume that the system is preceded by a "shaping filter" that has a white noise input, and generates the required multiple random inputs to the primary system.[9] The augmented system (original and shaping filter) then has a white noise input and a state vector that includes the original state variables as well as additional ones from the shaping filter. With a white noise input $w(t)$, there results a formula for the covariance matrix R like Eq. (9) but simpler, and also an expression for the correlation, i.e.,

$$O = AR + RA^T + BQB^T \tag{11}$$

$$C_{rr}(\tau) = Re^{-A\tau} \qquad \tau > 0$$

$$= e^{A\tau} R \qquad \tau < 0 \tag{12}$$

Here Q is the diagonal matrix of white noise intensities, so that

$$C_{ww}(\tau) = Q\delta(\tau) \tag{13}$$

The key to using the simpler formulas of the white noise case is to be able to generate the shaping filter that forms part of the augmented system. Holley and Bryson[10] give an approximate method for finding the filter matrices, and an exact method is given here in Appendix A.

Simulation of the Input

A direct method of finding the responses is of course to simulate the input random process by actually generating the required random functions (analog or digital) and using them as inputs to either a computer simulation or a physical realization of the system. This procedure would also be facilitated by using the shaping filter discussed above.

Equivalent Deterministic Input

A method was presented in Ref. 1 that applies only when there is a single input. It yields the mean-square response σ_r^2 as an integral of the response to an "equivalent deterministic input" $u(t)$. It utilizes the power-spectral density of the actual noise input $\phi_{nn}(\omega)$ to define the Fourier transform $U(\omega)$ of the function $u(t)$. When $u(t)$ is used as an input to the given (quiescent) system the response $y(t)$ yields the desired result via the equivalence

$$\int_{-\infty}^{\infty} y^2(t) \mathrm{d}t = \sigma_r^2 \tag{14}$$

This theorem has been used in the calculation of the response of airplanes to random atmospheric turbulence.[2-4]

Of the above methods, all but the last two require a mathematical model of the system. The two exceptions can be used when no model is available, although the system itself is, by applying the appropriate inputs and measuring the responses. Of these two, the last, the equivalent deterministic method, is much to be preferred from the standpoint of ease and cost of implementation. The drawback is that Ref. 1 treats only the case of a single input.

III. Multiple Deterministic Inputs

This paper is concerned exclusively with linear, asymptotically stable, time-invariant, causal (LASTIC) systems, a very common engineering model.

The zero-mean inputs are characterized by their correlation and spectrum matrices:

$$C_{nn}(\tau) \triangleq \langle n(t+\tau) n^T(t) \rangle \tag{15}$$

$$\Phi_{nn}(\omega) \triangleq \frac{1}{2\pi} \int_{-\infty}^{\infty} C_{nn}(\tau) e^{-i\omega\tau} \mathrm{d}\tau \tag{16}$$

By virtue of ergodicity, $\langle \rangle$ represents either time or ensemble average. Suppose now that the random functions $n(t)$ are inputs to the LASTIC system characterized by the transfer

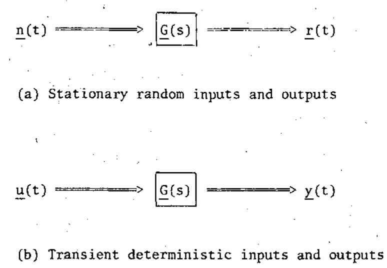

Fig. 1 Two types of inputs to the same LASTIC system [represented by the transfer function matrix, $G(s)$].

function matrix $G(s)$. The outputs $r(t)$ also have zero mean. The spectrum matrix associated with $r(t)$ is related to the input spectrum matrix by Eq. (1). The output correlation matrix is the inverse Fourier integral of ϕ_{rr}, i.e.,

$$C_{rr}(\tau)=\int_{-\infty}^{\infty} G(i\omega)\Phi_{nn}(\omega)G^H(i\omega)e^{i\omega\tau}\mathrm{d}\omega \tag{17}$$

This is the first of two pivotal results from LASTIC system theory.

In order to make the fundamental analogy to which attention is drawn in this paper, a second set of inputs $u(t)$ is applied to the same system (see Fig. 1). The corresponding outputs are denoted $y(t)$. Unlike the random inputs $n(t)$ of the last paragraph, the inputs $u(t)$ are deterministic and transient. The input-output relationship can be expressed thus:

$$Y(\omega)=G(i\omega)U(\omega) \tag{18}$$

Here, $Y(\omega)$ and $U(\omega)$ are, respectively, the Fourier transforms of the outputs $y(t)$ and the inputs $u(t)$. It is known from Parseval's theorem[7] (generalized to several variables) that

$$\int_{-\infty}^{\infty} y(t+\tau)y^T(t)\mathrm{d}t=\frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega)Y^H(\omega)e^{i\omega\tau}\mathrm{d}\omega \tag{19}$$

Therefore, from Eq. (18), the matrix of integral square outputs is

$$\int_{-\infty}^{\infty} y(t+\tau)y^T(t)\mathrm{d}t$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} G(i\omega)U(\omega)U^H(\omega)G^H(i\omega)e^{i\omega\tau}\mathrm{d}\omega \tag{20}$$

The analogy between this formula for deterministic, finite-energy signals and the formula, Eq. (17), for stationary random signals is the central point of this paper. Clearly, if one could write

$$UU^H=2\pi\Phi_{nn} \tag{21}$$

then the right-hand sides of Eqs. (17) and (20) would be identical for all G, and one could find the covariances C_{rr} by integrating the deterministic responses on the left-hand side of Eq. (20). Unfortunately, Eq. (21) can be shown to be possible only for a single input ($N=1$). This is because there are too few disposable functions in UU^H to match the number in ϕ_{nn}. Clearly what is needed is a modification of Eq. (21) that yields N^2 degrees of freedom on the left-hand side. This is achieved by considering a series of N inputs $u_k(t)$, $k=1,\ldots N$, whose Fourier transforms form the matrix

$$W\triangleq[U_1\ldots U_N] \tag{22}$$

For each input, Eq. (20) of course applies:

$$\int_{-\infty}^{\infty} y_k(t+\tau)y_k^T(t)\mathrm{d}t=\frac{1}{2\pi}\int_{-\infty}^{\infty} GU_kU_k^HG^He^{i\omega\tau}\mathrm{d}\omega \tag{23}$$

and the sum of Eq. (23) for all k is

$$\sum_{k=1}^{N}\int_{-\infty}^{\infty} y_k(t+\tau)y_k^T(t)\mathrm{d}t=\frac{1}{2\pi}\int_{-\infty}^{\infty} GWW^HG^He^{i\omega\tau}\mathrm{d}\omega \tag{24}$$

since from Eq. (22)

$$\sum_{k=1}^{N} U_kU_k^H=WW^H \tag{25}$$

We now compare Eq. (24) with Eq. (17) and note that for all G,

$$\sum_{k=1}^{N}\int_{-\infty}^{\infty} y_k(t+\tau)y_k^T(t)\mathrm{d}t=C_{rr}(\tau) \tag{26}$$

if

$$WW^H=2\pi\Phi_{nn} \tag{27}$$

Most commonly, Eq. (26) would be used to calculate only the diagonal elements of C_{rr} and then only for $\tau=0$.

Equation (27) yields a unique result for W if it is lower triangular with a real diagonal. Thus the pattern for W is

$$\begin{bmatrix} a_{11} & 0 & \ldots & 0 \\ a_{21}e^{i\theta_{21}} & a_{22} & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ a_{N1}e^{i\theta_{N1}} & a_{N2}e^{i\theta_{N2}} & \ldots & a_{NN} \end{bmatrix} \tag{28}$$

The form of Eq. (27) shows that a complex Choleski factorization is required. In other words, the Choleski square root of ϕ_{nn} is needed at each ω. A method for carrying out this computation for real numbers is given in Ref. 8. We found that it was relatively straight forward to extend it to complex ϕ_{nn}. The details are omitted in the interest of brevity. When $W(\omega)$ has been calculated, so that $U_1(\omega)\ldots U_N(\omega)$ are known, the appropriate input function sets $u_1(t)\ldots u_N(t)$ can be found by fast Fourier transform methods.

In general, of course W contains complex elements, but because ϕ_{nn} is Hermitian, it can be shown that the real parts of the $U_{ij}(\omega)$ are even functions of ω. It then follows that the inverse Fourier transform $u_{ij}(t)$ is real. It is the sum of two parts, one even in t [the inverse cosine transform of the real part of $U_{ij}(\omega)$] and one odd in t [the inverse sine transform of the imaginary part of $U_{ij}(\omega)$].

Relation for $\langle \dot{r}\dot{r}^T\rangle$

For some statistical analyses, as previously noted, the mean-squre of $\dot{r}_i$ may be required.[5] The theorem needed parallels Eq. (19). Since

$$F[\dot{y}(t)]=i\omega Y(\omega) \tag{29}$$

it follows from Eq. (19) that

$$\int_{-\infty}^{\infty} \dot{y}(t+\tau)\dot{y}^T(t)\mathrm{d}t$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} \omega^2G(i\omega)U(\omega)U^H(\omega)G^H(i\omega)e^{i\omega\tau}\mathrm{d}\omega \tag{30}$$

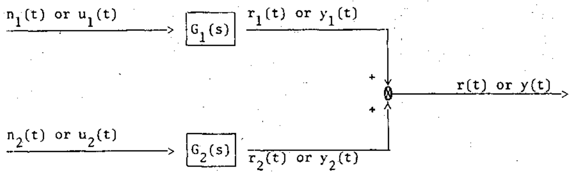

Fig. 2 Example system.

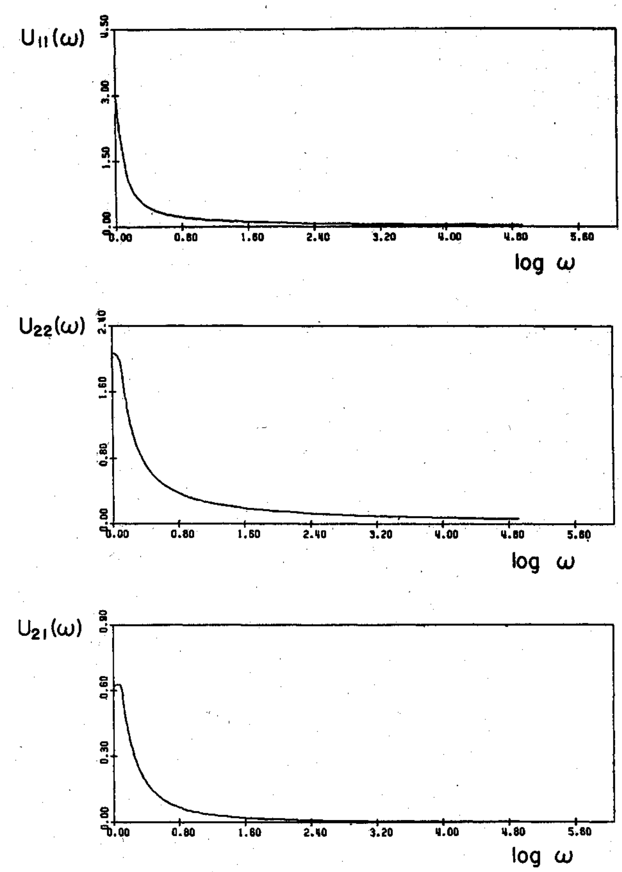

Fig. 3 Fourier transforms $U_{ij}(\omega)$.

But it is also true that

$$C_{\dot{r}\dot{r}}(\tau) = \int_{-\infty}^{\infty} \omega^2 G(i\omega)\Phi_{nn}(\omega)G^H(i\omega)e^{i\omega\tau}\mathrm{d}\omega \tag{31}$$

and hence it follows that for the set of N inputs previously described, so long as Eq. (27) holds

$$\sum_{k=1}^{N}\int_{-\infty}^{\infty} \dot{y}_k(t+\tau)\dot{y}_k^T(t)\,\mathrm{d}t = C_{\dot{r}\dot{r}}(\tau) \tag{32}$$

Thus the calculation of quantities such as $\langle \dot{r}_i^2 \rangle$, which entails evaluating diagonal terms of Eq. (32) for $\tau=0$, can proceed simultaneously with the calculation of terms in C_{rr}.

IV. Example

We now illustrate the method by an example related to flight in atmospheric turbulence since the spectrum functions used have a form commonly used to represent atmospheric turbulence (the Dryden spectrum[5])

$$\phi_{n_1n_1} = a/(1+b\omega^2)$$

$$\phi_{n_2n_2} = c(1+\mathrm{d}\omega^2)/(1+\sqrt{3}\mathrm{d}\omega^2)^2$$

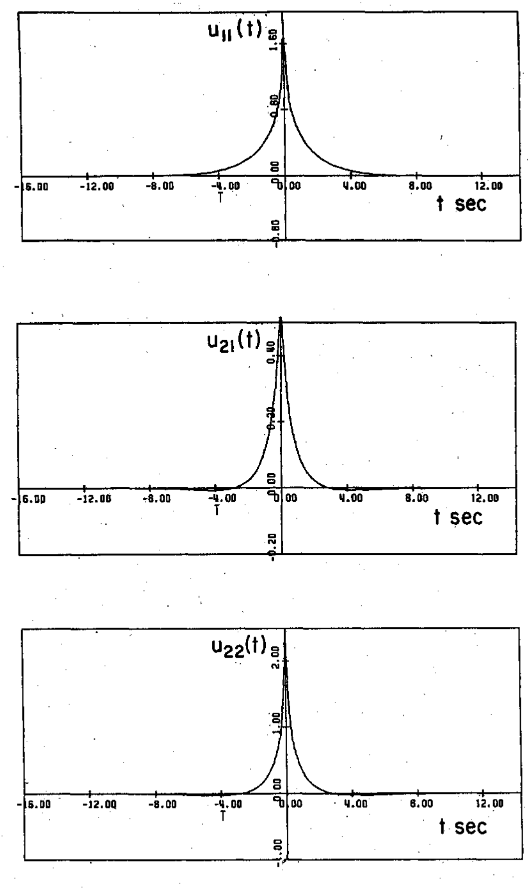

Fig. 4 Equivalent deterministic inputs.

$$\phi_{n_1n_2} = h[\phi_{n_1n_1}\phi_{n^{n_2}}/1+k\omega^2]^{1/2}$$

The constants have the following values:

$a=6.4972$ $b=7.5226$ $c=1.4225$

$d=2.0713$ $k=0.0753$ $h=0.5890$

The example system is characterized by two transfer functions (Fig. 2)

$G_1 = 2/(s^2+2\zeta_1\omega_1 s+\omega_1^2)$ $\omega_1 = 1r/s,$ $\zeta_1=0.15$

$G_2 = 5/(s^2+2\zeta_2\omega_2 s+\omega_2^2)$ $\omega_2 = 2r/s,$ $\zeta_2=0.30$

The Fourier transforms of the equivalent input time functions are shown in Fig. 3. Because the $\phi_{n_1n_2}$ used has no imaginary part, these functions are all real and even in ω.

The input time functions $u_{ij}(t)$ are shown in Fig. 4, and are also seen to be even in t. The responses $y_{ij}(t)$ to these inputs are shown in Fig. 5. The integrals of these responses were calculated according to Eqs. (26) and (32) for $\tau=0$, with the results shown in Table 1. The quantities shown in Table 1 relate to those shown in the figures as follows: For the spectral method

$$\sigma_r^2 = \langle r_1r_1\rangle + 2\langle r_1r_2\rangle + \langle r_2r_2\rangle$$

$$\sigma_{\dot{r}}^2 = \langle \dot{r}_1\dot{r}_2\rangle + 2\langle \dot{r}_1\dot{r}_2\rangle + \langle \dot{r}_2\dot{r}_2\rangle$$

where

$$\langle r_jr_k\rangle = \int_{-\infty}^{\infty}\Phi_{r_jr_k}(\omega)\mathrm{d}\omega \tag{33}$$

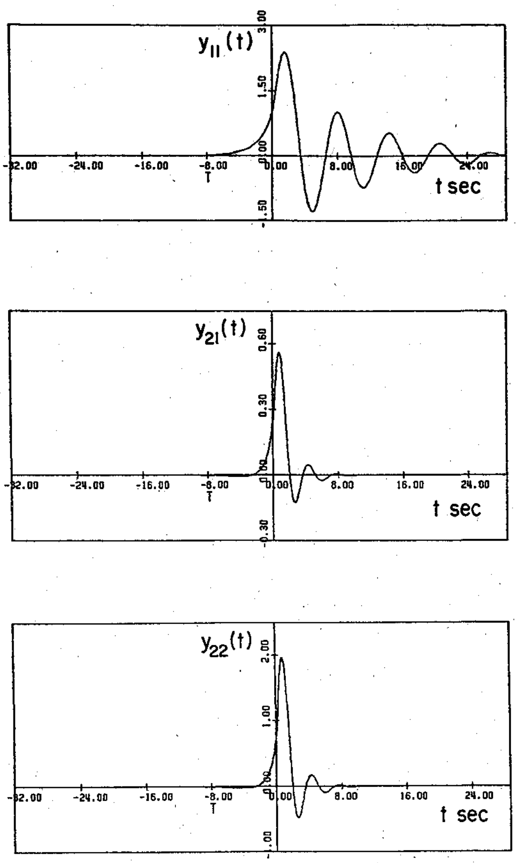

Fig. 5 Response to equivalent inputs.

Table 1 Comparison of results

Method	σ_r	$\sigma_{\dot{r}}$
Spectral		
Eqs. (33), (34)	4.559	3.954
Transient		
Eqs. (12), (20)	4.554	3.979

and

$$\langle \dot{r}_j \dot{r}_k \rangle = \int_{-\infty}^{\infty} \omega^2 \Phi_{r_j r_k}(\omega) \mathrm{d}\omega \tag{34}$$

where $\Phi_{r_j r_k}$ was calculated from Eq. (1).

For the transient method

$$\sigma_r^2 = C_{rr}(0) = \int_{-\infty}^{\infty} (y_{11}^2 + 2y_{21}y_{11} + y_{21}^2)\mathrm{d}t + \int_{-\infty}^{\infty} y_{22}^2 \mathrm{d}t$$

The corresponding expression for $\sigma_{\dot{r}}^2$ has $\dot{y}_{jk}$ replacing y_{jk} in the integrals.

The numerical values obtained by the two methods are seen to be in substantial agreement. Excluding the time taken to find $u_{jk}(t)$, the CPU time used for the transient method was 45% of that taken by the spectral method.

V. Concluding Remarks

The equivalent deterministic technique, originally presented in 1961 for a system with only one input, has been generalized to any number of inputs and outputs. A numerical example confirms that the results obtained for output covariances are the same as those obtained by spectral analysis.

When applied to situations where the equivalent input transients can be precalculated (i.e., situations with known forms of spectra), we think that appreciable savings of computing time may be realized over the corresponding conventional spectral analysis. This is the case when calculating aircraft response to atmospheric turbulence, since the form of the turbulence model is known. All the deterministic time functions, apart from amplitude and time-scaling factors, can then be predetermined and need be calculated only once for each size of airplane. When applied in situations where the system itself is available, but not an analytical model, it is our opinion that the use of equivalent deterministic inputs provides substantial time savings over the alternative methods, i.e., plant identification or generating a set of correlated random inputs with a prescribed correlation matrix.

Appendix: Shaping Filter

Suppose it is desired to have a set of N stationary random signals $\boldsymbol{n}(t)$ with prescribed spectral properties. That is, $\boldsymbol{\Phi}_{nn}(\omega)$ is specified. Or, equivalently, the correlation matrix $\boldsymbol{C}_{nn}(\tau)$ is specified. These signals are to be produced by passing white noise inputs through a 'signal-shaping' network. The white noise inputs are denoted $\boldsymbol{w}(t)$. They are assumed uncorrelated and to have constant spectral density of unit mangitude. That is,

$$\boldsymbol{\Phi}_{ww}(\omega) \equiv \boldsymbol{I} \tag{A1}$$

where $\boldsymbol{I}$ is the $N \times N$ unit matrix. Alternatively, the covariance matrix is

$$\boldsymbol{C}_{ww}(\tau) = \boldsymbol{I}\delta(\tau) \tag{A2}$$

The frequency response matrix for the signal shaping network is denoted $\boldsymbol{S}(\omega)$. We wish to find the $\boldsymbol{S}(\omega)$ that produces the desired $\boldsymbol{\Phi}_{nn}(\omega)$ at the output.

The relationship between $\boldsymbol{\Phi}_{nn}$ and $\boldsymbol{\Phi}_{ww}$ is the same as Eq. (1):

$$\boldsymbol{\Phi}_{nn}(\omega) = \boldsymbol{s}(\omega)\boldsymbol{\Phi}_{ww}(\omega)\boldsymbol{S}^H(\omega) \tag{A3}$$

which, in view of (A1), becomes

$$\boldsymbol{S}(\omega)\boldsymbol{S}^H(\omega) = \boldsymbol{\Phi}_{nn}(\omega) \tag{A4}$$

Thus the desired $\boldsymbol{S}(\omega)$ is the Choleski square root of the given spectral density matrix $\boldsymbol{\Phi}_{nn}(\omega)$.

It is clear from a comparison of Eqs. (7) and (A4) that, in fact,

$$\boldsymbol{W}(\omega) \equiv \sqrt{2\pi}\boldsymbol{S}(\omega) \tag{A5}$$

This shows another dimension to the analogy.

To complete the process of generating the shaping filter, a "realization" of $\boldsymbol{S}(\omega)$ is required. One method of carrying this out is given in Ref. 11.

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